Another Characterization of Expected Scott-Suppes Utility Representation*

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Abstract

This paper provides a new characterization of Expected Scott-Suppes Utility Representation (ESSUR). ESSUR combines the Expected Utility Representation with the Scott-Suppes Utility Representation (ESSUR). The latter represents semiorders that formalize preferences with intransitive indifferences. Dalkıran, Dokumacı, and Kara (2018) were the first to provide an axiomatic characterization of ESSUR. Surprisingly, their characterization does not involve any separability axiom, which is essential for numerical representations of preferences on uncountable sets. Candeal and Indurain (2010) were the first to provide a characterization of SSUR on uncountably infinite sets employing a semiorder separability axiom. In this paper, we identify the axioms required on top of those of Candeal and Indurain (2010) in order to obtain a linear Scott-Suppes utility representation, i.e., another characterization of ESSUR.

JEL Codes: C70, D01, D81
Keywords: Expected Utility, Intransitive Indifference, Scott-Suppes Utility Representation.

Beklenen Scott-Suppes Fayda Gösteriminin Başka Bir Karakterizasyonu

Özet

Bu makalede geçişken olmayan kayıtsızlıklara sahip tercihler fikrini bünyesinde barındıran yarı-sıralamalar yapısı altında Scott-Suppes fayda gösterimi ile Beklenen Fayda Gösterimi'ni ilişkilendiren Beklenen Scott-Suppes Fayda Gösterimi'nin yeni bir karakterizasyonu verilmektedir. Beklenen Scott-Suppes Fayda Gösterimi'nin ilk karakterizasyonu Dalkıran, Dokumacı ve Kara (2018) tarafından yapılmıştır. Dalkıran, Dokumacı ve Kara (2018)'nın karakterizasyonu yarı-sıralamalar için herhangi bir ayrılabilirlik aksiyomu kullanmamaktadır. Halbuki ayrılabilirlik aksiyomları sayılamaz sonsuzluktaki kümeler üzerindeki fayda gösterimlerinin temelinde yer almaktadırlar. Sayılamaz sonsuzluktaki kümeler üzerinde yarı-sıralamalar için bir ayrılabilirlik aksiyomu kullanarak Scott-Suppes Fayda Gösterimi karakterizasyonunu elde eden ilk çalışma Candeal ve Indurain (2010)'dır. Bulduğumuz karakterizasyon Candeal ve Indurain (2010)'un karakterizasyonunu temel olarak alıp hangi aksiyomlar ilave edildiğinde bir Beklenen Scott-Suppes fayda gösterimi elde edilebilir sorusunu cevaplamaktadır.

JEL Sınıflandırması: C70, D01, D81
Anahtar Kelimeler: Beklenen Fayda, Geçişken Olmayan Kayıtsızlıklar, Scott-Suppes Gösterimi.

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Utility theory lies at the heart of the modern economic theory. Decision-makers in many economic interactions are modeled as utility maximizers. Albeit their tractability, utility representations come with a cost because the resulting decision-makers are de facto rational. That is, their preferences can be represented with complete and transitive preferences. However, these axioms have been heavily criticized. In this paper, we highlight such a critique by focusing on an implausible implication of the transitivity axiom, which is defined as follows: if x is at least as good as y and y is at least as good as z, then x is at least as good as z. Here, the concept of 'being at least as good as' is generally partitioned into two different preference relations: a strict preference relation and an indifference relation. An interpretation of the indifference relation is the lack of a strict preference: If an individual does not strictly prefer an alternative x over another alternative y and s/he does not strictly prefer y over x, then s/he is said to be indifferent between x and y.

A strict preference relation is transitive if an individual strictly prefers x to y, and strictly prefers y to z, then s/he strictly prefers x to z. On the other hand, if an individual is indifferent between x and y and indifferent between y and z, then it may seem to be reasonable to assume that the individual is indifferent between x and z. If this is the case, we say that the indifference relation is transitive as well.

Many economists argue that rational choice implies that the strict preference relation must be transitive. On the other hand, it is not clear whether rationality imposes the transitivity of the indifference relation. The limited perception of humankind and empirical unresponsiveness of individuals to small changes seem to support the intransitivity of the indifference relation. That is, an individual may be indifferent between x and y, and s/he may be indifferent between y and z, but s/he does not have to be indifferent between x and z.

The idea of intransitivity of indifferences is not only a topic of interest in economics. Indeed, it has also been analyzed in philosophy, physics, psychology, and psychophysics. For example, the existence of 'intransitive indifference' is related to the concept of 'vagueness' in philosophy. The sorites paradox is a well-known example:

> 1000000 grains of sand make a heap.
>
> If 1000000 grains of sand make a heap, then 999999 grains of sand do.
>
> If 999999 grains of sand make a heap, then 999998 grains do.
>
> ...
>
> If 2 grains of sand make a heap, then 1 grain does. 1 grain of sand makes a heap.

The morale of the sorites paradox is that an individual may not differentiate between two very close quantities. That is, for an individual to recognize the difference, they must differ more than some threshold. The Weber-Fechner law highlighted this observation in the field of psychophysics as follows: The actual change and perceived change of a stimulus may not necessarily coincide because some changes in some ranges may be unnoticeable, just like the case of intransitive indifferences.

When an individual strictly prefers an alternative over another and his/her preferences exhibit intransitive indifferences, s/he behaves as if these two alternatives differ from each other more than some threshold. Under the standard assumption of transitive indifference, this threshold can be considered as zero. However, the threshold level may be non-zero under intransitive indifference. To illustrate, one can be indifferent between sleeping at 10:00 PM and 11:00 PM and be indifferent between sleeping at 11:00 PM and 12:00 AM. However, s/he might strictly prefer sleeping at 10:00 PM rather than sleeping at midnight. In this example, the threshold level for such an individual can be thought of as an hour.[1]

The idea of intransitive indifferences has been in the literature since the 19th century (see Weber (1834)). In the 20th century, Luce (1956), introduced the concept of 'semiorder' to capture the idea of intransitive indifferences.

> It is certainly well known from psychophysics that if 'preference' is taken to mean which of two weights a person believes to be heavier after hefting them, and if 'adjacent' weights are properly chosen, say a gram difference in a total weight of many grams, then a subject will be indifferent between any two adjacent weights. If indifference were transitive, then he would be unable to detect any weight differences, however great, which is patently false. (Luce, 1956)

Scott and Suppes (1958) provide a utility representation for preferences that exhibit intransitive indifference –represented as semiorders– on finite sets. This representation implies that the utility difference between two alternatives must be more than some threshold level for an individual to have a strict preference between these alternatives. This is aligned with the intuition that a strict preference can arise only when the difference is more than some threshold.[2]

On the other hand, semiorders on infinite sets are not always representable in the sense of Scott-Suppes. Beja and Gilboa (1992) provide necessary and sufficient conditions for a semiorder to have a Scott-Suppes type of utility representation on countably infinite sets. Gensemer (1987) provides an axiomatic characterization for a continuous Scott-Suppes type of representation. Recently, Candeal and Indurain (2010) present a characterization of Scott-Suppes representability of a semiorder on an uncountably infinite set.

Fishburn (1968) studies preferences with intransitive indifference under risk. He shows that the sure-thing principle, an essential axiom of expected utility, is incompatible with intransitive indifference and lists the characterization of a Scott-Suppes type of expected utility representation as an open problem. More recently, Dalkıran, Dokumacı, and Kara (2018) provide an answer to this open problem by presenting such an axiomatic characterization.

The characterization of Candeal and Indurain (2010) points out two properties, namely regularity and separability, which are necessary and sufficient for a Scott-Suppes utility representation of a semiorder on an uncountably infinite set. Even though Dalkıran, Dokumacı, and Kara (2018) provide a (linear) Scott-Suppes utility representation on an uncountably infinite set, their characterization does not utilize any separability axioms.[3] This is surprising because separability axioms are essential to any utility representation on an uncountably infinite set. This begs an answer to the question of whether one can provide a characterization of the expected Scott-Suppes utility representation by employing a separability axiom. In this paper, we provide such a characterization by taking the axioms provided by Candeal and Indurain (2010) as given and identifying what additional axioms are required to achieve another characterization of Expected Scott-Suppes Utility Representation (henceforth ESSUR).

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[1] We note that the sleeping time example we provide is similar in nature to Luce's (1956) famous coffee-sugar example.

[2] See Fishburn (1968) and Fishburn (1985) for more on semiorders and intransitive indifference. Gilboa and Lapson (1995) argue that the standard weak order approach is not an appropriate approximation for preferences with intransitive indifference.

[3] The separability property is a key axiom to obtain a numerical representation of preferences with intransitive indifference. For more on separability axioms, see Bosi, Candeal, Indurain, Oloriz, and Zudaire (2001) and Candeal, Indurain, Garcia, and Indurain (2012).

Preliminaries

We first introduce definitions, concepts, and axioms from the literature that will be frequently used in this study.

As our aim is to provide another characterization of ESSUR, we restrict ourselves to a set of lotteries over a finite set. Let $X = \{x_1, x_2, x_3, ..., x_n\}$ denote a set with $n \in N$ alternatives. A lottery on X is a list $p = \{p_1, p_2, p_3, ..., p_n\}$ such that $\sum p_i = 1$ and for each $i \in \{1, 2, 3, ..., n\}$, we have $p_i \geq 0$ where x_i occurs with probability p_i. That is, L is the set of all (objective) lotteries on the finite set X.

Let $R \subseteq L \times L$ be a reflexive binary relation on L.[4] We write $x\ R\ y$ in lieu of $(x, y) \in R$. We define the **strict part** of R, denoted by P, as $x\ P\ y$ if $x\ R\ y$ and $\neg(y\ R\ x)$. Similarly, we define the **indifference part** of R, denoted by I, as $x\ I\ y$ if $x\ R\ y$ and $y\ R\ x$. Observe that R is the union of the binary relations, P and I, on the set L, i.e., $R = P \cup I \subseteq L \times L$.

We assume that P and I induced by R on L satisfy trichotomy: Only one of $x\ I\ y$, $x\ P\ y$ or $y\ P\ x$ holds. Furthermore, it is straightforward to see that under trichotomy, we have $x\ R\ y$ if $\neg(y\ P\ x)$.

Definition 1. Given a reflexive binary relation R on L that satisfies trichotomy, we define the following auxiliary binary relations on L: For each $x, y \in L$,

- $x\ P_0\ y$ if there exists $z \in L$ such that $x\ R\ z\ P\ y$ or $x\ P\ z\ R\ y$,
- $x\ R_0\ y$ if $\neg(y\ P_0\ x)$,
- $x\ I_0\ y$ if $x\ R_0\ y$ and $y\ R_0\ x$.

Definition 2. R is an **interval order** on L if

I1. I is reflexive,

I2. for each $x, y \in$ L, exactly one of $x\ P\ y$, $y\ P\ x$ or $x\ I\ y$ holds,

I3. for each $x, y, z, t \in L$, $x\ P\ y$ and $z\ P\ t$ imply $x\ P\ t$ or $z\ P\ y$.

Definition 3. R is a **semiorder** on L if

S1. I is reflexive,

S2. for each $x, y \in L$, exactly one of $x\ P\ y$, $y\ P\ x$ or $x\ I\ y$ holds,

S3. for each $x, y, z, t \in L$, $x\ P\ y$ and $z\ P\ t$ imply $x\ P\ t$ or $z\ P\ y$,

S4. for each $x, y, z, t \in L$, $x\ P\ y$ and $y\ P\ z$ imply $x\ I\ t$ imply $t\ P\ z$.

It is straightforward to see that every semiorder is an interval order; however, the inverse is not always true.

[4] R is said to be reflexive if for each $x \in L$, we have xRx.

Definition 4. Let R be a binary relation on L, u: $L \rightarrow$ R be a function, and $k \in R_{++}$. (u, k) is a **Scott-Suppes utility representation** of R if for each $x, y \in L$, $x\, P\, y$ if and only *if* $u(x) > u(y) + k$.

If the preferences of an individual can be represented by a Scott-Suppes utility representation as described above, $k \in R_{++}$ can be interpreted as the threshold level of utility difference for this individual to break the (intransitive) indifference. It follows that, if the difference in terms of utility is less than or equal to k between two alternatives, the individual is indifferent between these alternatives. On the other hand, for an individual to have a strict preference, the utility difference between two alternatives must be more than the threshold level $k \in R_{++}$.

Definition 5. A function u: $L \rightarrow$ R is **linear** if for each $x, y \in L$ and for each $\alpha \in (0, 1)$, we have $u(\alpha x + (1- \alpha)\, \mathrm{y}) = \alpha.u(x) + (1- \alpha).u(y)$.

Linearity of utility function is essential for expected utility representation.

The Axioms Employed in Our Characterization

Below, we define the axioms that will be used in our main result, i.e., in our characterization of ESSUR.

Definition 6. A semiorder R on L is **semiorder-separable** if there is a countable subset $D \subseteq L$ with the following property: for every $x, y \in L$ such that $x\, P\, y$, there are $d_1, d_2 \in D$ such that $x\, P\, d_1\, R_0\, y$ and $x\, R_0\, d_2\, P\, y$.

Definition 7. A semiorder R on L is **strongly separable** if there is a countable subset $D \in L$ with the following property: for every $x, y \in L$ such that $x\, P\, y$, there are $d_1, d_2 \in D$ such that $x\, P\, d_1\, R\, d_2\, P\, y$.

Candeal and Indurain (2010) show that semiorder separability is a necessary condition for a Scott-Suppes utility representation.

On the other hand, strong separability of R is introduced by Chateauneuf (1987) and, it is a necessary condition for continuous Scott-Suppes representation as shown in Gensemer (1987). We note that strong separability implies semiorder separability and the inverse is not always true.

Definition 8. A binary relation R on L is **regular** if there is no $x, y \in L$ and no sequences (x_n), $(y_n) \in L^{\mathrm{N}}$ such that for each $n \in N$, we have $x\, P\, x_n$ and $x_{n-1}\, P\, x_n$ or for each $n \in N$, we have $y_n\, P\, y$ and $y_n\, P\, y_{n-1}$. That is, the set L has no infinite up- or down- chains with regards to P with an upper or a lower bound, respectively.

Regularity is a necessary condition for both Scott-Suppes utility representation and ESSUR. The former is proved by Candeal and Indurain (2010) and, the latter is proved by Dalkıran, Dokumacı, and Kara (2018).

Definition 9. A reflexive binary relation R on L is **mixture-symmetric** if for each $x, y \in L$ and each $\alpha \in [0, 1]$, $x\, I\, \alpha x + (1\text{-}\alpha)y$ implies $y\, I\, \alpha y + (1\text{-}\alpha)x$.

Mixture symmetry is introduced by Nakamura (1988) for a characterization of an expected utility representation for interval orders. This axiom is also used by Dalkıran, Dokumacı, and Kara (2018) in their characterization of ESSUR. Mixture symmetry is essential for the linearity of the utility function representing a semiorder or an interval order.

Definition 10. R_0 on L is **continuous** if for each y P L, the sets

$$UC(y) := \{x \in L : x\, R_0\, y\} \text{ and } LC(y) := \{x \in L : y\, R_0\, x\}$$

are closed with respect to the Euclidean metric on R.

On the other hand, R_0 on L is **mixture-continuous** if for each x, y, z P L, the sets

$$UMC(y; x, z) := \{\alpha \in [0, 1] : [\alpha x + (1-\alpha)z]\ R_0\ y\}$$
and
$$LMC(y; x, z) := \{\alpha \in [0, 1] : [y\ R_0\ \alpha x + (1-\alpha)z]\}$$

are closed with respect to the Euclidean metric on R.

Definition 11. R_0 on L satisfies the **midpoint indifference axiom** if for each $x, y, z \in L$,

$x\ I\ y$ implies $1/2x + 1/2z\ I_0\ 1/2y + 1/2z$.

Mixture continuity and midpoint indifference are necessary and sufficient conditions for an expected utility representation of a weak-order as shown in Hernstein and Milnor (1954). We note that mixture continuity is a weaker condition than the standard continuity axiom, i.e., continuity of a weak order implies mixture continuity of the same weak order.[5]

Representation Theorems

Representation Theorems from the Literature

In this section, we introduce several results and representation theorems from the literature. The first theorem presents necessary and sufficient conditions for an expected Scott-Suppes utility representation.

We emphasize that even though this theorem characterizes a Scott-Suppes utility representation on an uncountable set, it does not utilize any type of 'separability' axiom. However, separability of a semiorder or an interval order is an essential axiom for utility representations under intransitive indifference.

At this point, we would like to emphasize that the goal of this study can be thought of as identifying a characterization of Expected Scott-Suppes Utility Representation (ESSUR) using a separability axiom.

Theorem 1. *(Dalkıran, Dokumacı, and Kara (2018)) Let R be a non-trivial semiorder on L.*

- *R is regular and mixture-symmetric,*
- *R_0 is mixture-continuous and satisfies the midpoint indifference axiom,*
- *for each $x, y \in L$, if $x\ P\ y$, then there exists $z \in L$ such that $x\ I\ z$ and for each $t \in L$, we have $z\ P_0\ t$ implies $x\ P\ t$.*

if and only if there exists a linear function $u: L \rightarrow$ R and $k \in R_{++}$ such that (u, k) is a Scott-Suppes utility representation of R.

[5] See Inoue (2010) for further details.

We note that the theorem above is the first characterization of ESSUR in the literature.

The next theorem is a result that shows necessary and sufficient conditions for Scott-Suppes representation of a semiorder on uncountable-infinite sets.

Theorem 2. *(Candeal, Indurain (2010)) Let R be a non-trivial semiorder on L. Then, the following are equivalent:*

- *R is Scott-Suppes representable.*
- *R is a regular and semiorder-separable semiorder.*

It is noteworthy to mention that Candeal and Indurain's (2010) characterization does not guarantee that u is continuous and/or linear. Therefore, it is not a characterization of ESSUR.

Next, we present a characterization of a continuous Scott-Suppes utility representation of a semiorder as given by Gensemer (1987). To provide this result, we need the following:

Definition 12. A semiorder R on L is symmetrically regular[6] on L if the following hold:

- If $x, y \in L_M$ and if there exists $z \in L$ such that $x\ R\ z\ P\ y$, then there exists t P L such that $x\ P\ t\ R\ y$, and
- If $x, y \in L_m$ and if there exists $z \in L$ such that $x\ P\ z\ R\ y$, then there exists t P L such that $x\ R\ t\ P\ y$.

where L_M is the set of non-minimal elements and L_m is the set of non-maximal elements with respect to P.

The next definition is also from Gensemer (1987).

Definition 13. A semiorder R on L is normal if the following hold:

- If $L_M \neq \emptyset$ and $L\text{-}L_M \neq \emptyset$, then there exists $x \in L_M$ and $y \in L - L_M$ such that $y\ R\ x$.
- If $L_m \neq \emptyset$ and $L\text{-}L_m \neq \emptyset$, then there exists $x \in L_M$ and $y \in L - L_m$ such that $x\ R\ y$.
- If $x \in W$, then there exists $y \in L$ such that $x\ P^*\ y\ R\ x$, where $W = L_M \cup L_m$, i.e., W is the set of elements which are neither minimal nor maximal elements in L.[7]

Normality axiom prevents isolation of an element in an indifference set whenever P is transitive.

We are now ready to present the aforementioned characterization theorem for a continuous Scott-Suppes utility representation:

Theorem 3. *(Gensemer (1987)) Let R be a non-trivial semiorder on L. Then,*

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[6] $x\ P^*\ y\ R\ x$ if there exists $z \in L$ such that $x\ R\ z\ P\ y$. Similarly, $xP^{**}y$ if there exists $t \in L$ such that $x\ P\ t\ R\ y$.

[7] This axiom is introduced by Gensemer (1987). However, Gensemer refers to this axiom simply as 'regularity'. To prevent confusion, we renamed it as 'symmetrical regularity' in this study.

- *R is strongly separable,*
- *R is symmetrically regular,*
- *R is normal*
- *R_0 is continuous,*

if and only if there exists a continuous function u: $L \rightarrow \mathrm{R}$ *and* $k \in R_{++}$ *such that (u, k) is a Scott-Suppes utility representation of R.*

A New Characterization of Expected Scott-Suppes Utility Representation

Before moving to the main result, we introduce some results and observations that will be building blocks of the proof of our main theorem.

We start with a relatively well-known result in the literature:

Lemma 1. If R is a semiorder, then for any $x, y, z, t \in L$, then

- $x\ P\ y\ I\ z\ P\ t \Rightarrow x\ P\ t$,
- $x\ P\ y\ P\ z\ I\ t \Rightarrow x\ P\ t$,
- $x\ I\ y\ P\ z\ P\ t \Rightarrow x\ P\ t$,
- $x\ P\ y\ R\ z\ P\ t \Rightarrow x\ P\ t$,
- $x\ P\ y\ P\ z\ R\ t \Rightarrow x\ P\ t$,
- $x\ R\ y\ P\ z\ P\ t \Rightarrow x\ P\ t$.

Proof. For the proof of this lemma, see either Bridges (1983) or Aleskerov, Bouyssou, and Monjardet (2007). □

Lemma 2. If a semiorder R on L is semiorder-separable, regular and satisfies mixture-symmetry, then R is normal.

Proof. First, recall that R is normal if the following hold:

- If x is neither a minimal nor a maximal element in L, then there exists $y \in L$ such that $x\ P^*\ y\ R\ x$. *(N1)* Equivalently, there exist y and t such that $x\ I\ t\ P\ y\ I\ x$.
- If the set of non-minimal elements and the set of minimal elements in L are non-empty, then there exists an element, x in the set of non-minimal elements and y in the set of minimal elements such that $y\ R\ x$. *(N2)*
- If the set of non-maximal elements and the set of maximal elements in L are non-empty, then there exists an element, x in the set of non-maximal elements and y in the set of maximal elements such that $x\ R\ y$. *(N3)*

Now observe that by Theorem 3.6 of Candeal and Indurain (2010), R has a Scott-Suppes representation (u, k) such that $x\ P\ y$ if and only if $u(x) > u(y) + k$.

We first prove that R satisfies *(N2)*. The proof of the fact that R satisfies *(N3)* is similar.

Suppose that R does not satisfy *(N2)*:

Then, for any non-minimal element $x \in L_M$ and any minimal element $y \in L\text{-}L_M$, we have $x\ P\ y$. Hence, $u(x) > u(y) + k$. Let $u^* = sup\{u(y)|y \in L\text{-}L_M\}$ and $u^{**} = inf\{u(x)|x \in L_M\}$. Then it follows that $u^{**} \geq u^* + k$. For $\varepsilon > 0$, take $x_0 \in L_M$ such that $u(x_0) < u^{**} + \varepsilon$ and take $y^* \in L\text{-}L_M$ such that $u(y^*) > u^* - \varepsilon$.

Consider the elements in L of the form $\alpha x_0+(1-\alpha)y^*$ for some $\alpha \in [0, 1]$. If $\alpha x_0+(1-\alpha)y^*$ is a minimal element, i.e., $\alpha x_0+(1-\alpha)y^* \in L - L_M$, then $\alpha x_0+(1-\alpha)y^*\ I\ y^*$ and hence, by mixture-symmetry, $(1\text{-}\alpha)x_0+1\text{-}\ \alpha y^*\ I\ x$. Therefore, $(1\text{-}\alpha)x_0+1\text{-}\ \alpha y^*$ is a non-minimal element, i.e., $(1\text{-}\alpha)x_0+1\text{-}\ \alpha y^* \in L_M$.

Next, consider $0.5x_0 + 0.5y^* \in L$. Observe that $0.5x_0 + 0.5y^*$ cannot be a minimal element in L since otherwise $0.5x_0+0.5y^* I\ y^*$ would imply $0.5x_0+0.5y^* I\ x$ by mixture symmetry. But then $0.5x_0 + 0.5y^*$ is non-minimal, a contradiction. On the other hand, we cannot have $0.5x_0+0.5y^* I\ x$ because otherwise, $0.5x_0+0.5y^* I\ x_0$ is non-minimal but also, by mixture symmetry, $0.5x_0+0.5y^*\ I\ y^*$, and hence a minimal element, a contradiction. Therefore, by trichotomy, we must either have $0.5x_0+0.5y^* P\ x_0$ or $x_0\ P\ 0.5x_0+0.5y^*$. Let $x_1 = 0.5x_0 + 0.5y^*$ and consider $x_2 = 0.5x_1 + 0.5y^* \in L$. A similar argument implies that x_2 is non-minimal with either $x_2\ P\ x_1$ or $x_1\ P\ x_2$. Continuing in this fashion gives us an infinite downward chain such that y° is a lower bound, i.e. $(y_n) \in L^{\mathrm{N}}$ with $y^n\ P\ y_{n+1}$ and $y_n\ P\ y^*$ for all $n \in \mathrm{N}$. This contradicts regularity. Thus, (N2) and (N3) hold.

Next, we prove that R satisfies (N1). Suppose that R does not satisfy (N1). Then, for any $x \in W = L_M \cap L_m$, there does not exist $y \in L$ such that $x\ P^* y\ R\ x$. Thus, for any $x \in W$ and $y \in L$, we have $\neg(x\ P^*\ y\ R\ x)$. This implies for any $x \in W$ and $y \in L$, either $\neg(x\ P^*\ y)$ or $\neg(y\ R\ x)$.

If for any $x \in W$ and $y \in L$, $\neg(x\ P^*\ y)$, then, by definition of P^*, there does not exist $z \in L$ such that $x\ R\ z\ P\ y$. That is, for any $z \in L$, we have $\neg(x\ R\ z\ P\ y)$. Taking $z = x$ implies that we cannot have $x\ P\ y$ for any $y \in L$. This implies that x is a minimal element in L, i.e., $x \in L - L_M$, a contradiction.

On the other hand, if for any $x \in W$ and $y \in L$ we have $\neg(y\ R\ x)$, then for any x, we have $x\ P\ y$ for all $y \in L$. Therefore, x is a maximal element in L, i.e., $x \in L - L_m$, a contradiction. Hence, R satisfies (N1) as well.

Therefore, we can conclude that if a semiorder R on L is semiorder-separable, regular and satisfies mixture-symmetry, then R is normal. □

To provide our next result, we need the following definitions:

Definition 14. A semiorder R on L is **full** if for every $x, y \in L$ such that $x\ P\ y$, there are $a, b \in L$ such that $x\ P\ a\ R\ b\ P\ y$.

We note that the fullness axiom has a very similar structure to the strong separability. The only difference is that the fullness condition does not require the existence of a countable subset of L.

Definition 15. An interval order R on L is **interval order-separable** if there is a countable subset $D \subseteq L$ with the following property: for every $x, y \in L$ such that $x\ P\ y$, there is $d \in D$ such that $x\ R^*\ d\ P\ y$.[8]

Lemma 3. (Bosi, Candeal, Indurain, Oloriz, and Zudaire (2001)) The following are equivalent:

- R is strongly separable,
- R is interval order-separable and full.

Proof. See Bosi, Candeal, Indurain, Oloriz, and Zudaire (2001). □

Candeal, Estevan, Garcia, and Indurain (2012) state that whenever R is a semiorder that satisfies the regularity axiom, then R is interval order-separable if and only if R is semiorder-separable. Because we work with a semiorder that satisfies the regularity axiom, we have the following immediate result:

Lemma 4. If R is a semiorder that satisfies the regularity axiom, then the following are equivalent:

- R is strongly separable.
- R is semiorder-separable and full.

Proof. The proof follows from Theorem 4.7 of Candeal, Estevan, Garcia, Indurain (2012). □

We know from Candeal and Indurain (2010) that when a semiorder R is semiorder-separable and regular, then R has a Scott-Suppes utility representation. Yet, we do not know whether the corresponding utility function is continuous.

The next theorem shows that the set of axioms we work with are sufficient for the existence of a continuous Scott-Suppes utility representation.

Theorem 4. *Let R be a non-trivial semiorder on L. If R is regular, separable and mixture-symmetric and R_0 is continuous and satisfies the midpoint indifference axiom, then there exists a continuous function u: $L \to \mathbb{R}$ and $k \in \mathbb{R}_{++}$ such that (u,k) is a Scott-Suppes utility representation of R.*

Proof. As stated in Theorem 3.1.3., Gensemer (1987) shows that if R is strongly separable, normal, symmetrically regular, and R_0 is continuous, then there exists a continuous function. So, if the axioms we use in our main theorem cover these axioms, we are done. We note that the axioms we use in our main theorem are as follows: R is semiorder-separable, regular, mixture-symmetric, and R_0 is continuous and satisfies the midpoint indifference axiom.

By Candeal and Indurain (2010), Theorem 3.1.2., we know that R is Scott-Suppes representable since the semiorder R is regular and semiorder-separable.

On the other hand, Bosi, Candeal, Indurain, Oloriz, and Zudaire (2001), as stated in Lemma 3, shows that R is strongly separable if and only if R is interval order-separable and full.

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[8] $x\ R^*\ y$ if $\neg(y\ P^*\ x)$ *and* $x\ P^*\ y$ if there exists $z \in L$ such that $x\ R\ z\ P\ y$.

Furthermore, Candeal, Estevan, Garcia, and Indurain (2012) shows that when R is regular, as stated in Lemma 4, R is strongly separable if and only if R is semiorder order-separable and full.

That is, to show that R is strongly separable, it is enough to show that R is full: Since R is semiorder-separable, there exists a countable subset $D \subseteq L$ such that defines its semiorder-separability, i.e., there exist $x,y \in L$ with $x\ P\ y$, there are $d_1,d_2 \in D$ such that $x\ R_0\ d_1\ P\ y$ and $x\ P\ d_2\ R_0\ y$. Observe that if $x\ P\ d_1$, then $x\ P_0\ d_1$ and similarly if $d_2\ P\ y$, then $d_2\ P_0\ y$. By definition of P_0, there exist $a,b \in L$ with $a\ R\ b$ such that $x\ P\ a\ R_0\ y$ and $x\ R_0\ b\ P\ y$. Under trichotomy, we have either $x\ R\ y$ or $x\ R\ y$, or both. Without loss of generality, assume that $x\ R\ y$. Then, we have $x\ P\ a\ R\ b\ P\ y$. Thus, R is full. Therefore, R is strongly separable.

Since, by Lemma 2 above, we also know that the semiorder R is normal. In order to finish to proof, it remains to be shown that R is symmetrically regular, i.e., if $x, y \in L_M$ where L_M is the set of elements which are not minimal in L and $x\ P\ z\ R\ y$, then there exists $t \in L$ such that $x\ R\ t\ P\ y$; and if $x, y \in L_m$ where L_m is the set of elements which are not maximal in L and $x\ R\ z\ P\ y$, then there exists $t \in L$ such that $x\ P\ t\ R\ y$.

Since R is semiorder-separable and full under our axioms, it is easy to see that R is symmetrically regular. For the sake of completeness, if R is semiorder-separable and full, then there are d_1, $d_2 \in L$ such that $x\ P\ d_1\ R_0\ y$ and $x\ R_0\ d_2\ P\ y$. By the existence of Scott-Suppes utility representation, the former implies $x\ P\ d_1\ Ry$ and the latter implies $xRd_2\ P\ y$. Therefore, R is symmetrically regular.

Thus, by Gensemer (1987), there exists a continuous u: $L \rightarrow \mathrm{R}$ and $k \in R_{++}$ such that (u, k) is a representation of R. □

We are now ready to present our main result, which provides a new characterization of ESSUR.

Theorem 5. *Let R be a non-trivial semiorder on L.*

- *R is regular, semiorder-separable and mixture-symmetric,*
- *R_0 is continuous and satisfies the midpoint indifference axiom*

if and only if there exists a linear function u: $L \rightarrow \mathrm{R}$ and $k \in R_{++}$ such that (u, k) is a Scott-Suppes utility representation of R.

Proof. First, we show that if a non-trivial semiorder R on L is regular, semiorder-separable and mixture-symmetric, and the corresponding R_0 is continuous and satisfies the midpoint indifference axiom, then there exists an expected Scott-Suppes representation of R. Observe that, by Theorem 3.2.1, we already know that there exists a continuous utility function u: $L \rightarrow \mathrm{R}$ *and* $k \in R_{++}$ *such that* (u, k) is a (continuous) Scott-Suppes representation of R.

We are left to show that there exists a linear utility function $\hat{u}$: $L \rightarrow \mathrm{R}$ *and* $\hat{k} \in R_{++}$ such that $(\hat{u}, \hat{k})$ is an expected Scott-Suppes representation of R.

To finish the proof, we employ the characterization of Dalkiran, Dokumaci, and Kara (2018). Observe that, as stated in Theorem 3.1.1, the difference of our axioms when compared to Dalkiran, Dokumaci, and Kara (2018) are as follows: We have R is semiorder-separable, R_0 is continuous instead of mixture-continuous, and finally, we do not have the existence of maximal indifference elements. Since R_0 being continuous implies R_0 is mixture-continuous (see Inoue (2010)), it is enough to show the existence of the maximal indifference elements.

Observe that L is compact with respect to the standard Euclidean metric and we note that by Heine-Borel Theorem, a subset of Euclidean space is compact if it is closed and bounded. Therefore, any closed subset of L is compact. Since u: $L \to$ R is continuous and represents R_0, then the Extreme Value Theorem implies the existence of maximal indifference elements, as desired: for each x, $y \in L$, if $x\ P\ y$, then there exists $z \in L$ such that $x\ I\ z$ and for each $t \in L$, we have $z\ P_0\ t$ implies $x\ P\ t$.

To sum up if R is a non-trivial semiorder on L and

- R is regular, semiorder-separable and mixture-symmetric,
- R_0 is continuous and satisfies the midpoint indifference axiom, then, R satisfies the following:
- R is regular and mixture-symmetric,
- R_0 is mixture-continuous and satisfies the midpoint indifference axiom,
- for each x, $y \in L$, if $x\ P\ y$, then there exists $z \in L$ such that $x\ I\ z$ and for each $t \in L$, we have $z\ P_0\ t$ implies $x\ P\ t$.

Therefore, by Dalkiran, Dokumaci, and Kara (2018), there exists an ESSUR of R, as desired.

Next, we need to show that ESSUR implies the axioms listed. Let (u, k) be an ESSUR of R. It follows Candeal, Indurain (2010) that R is semiorder-separable and regular. It also follows from Dalkıran, Dokumacı, and Kara (2018), R is a non-trivial, regular and mixture symmetric semiorder and, R_0 is mixture-continuous and satisfies midpoint indifference axiom. The only axiom left to show is that R_0 is continuous. This follows from the fact u: $L \to$ R represents the weak-order R_0. Furthermore, u is linear and hence continuous. Since upper-contour and lower-contour sets are inverse images of closed sets with respect to u and u is continuous, then R_0 is continuous as well. This finishes the proof. □

Independence of the Axioms

We know from Candeal and Indurain (2010) that semiorder-separability and regularity of R are mutually independent axioms. Similarly, we also know from Dalkıran, Dokumacı, and Kara (2018) that mixture continuity and midpoint indifference of R_0, and regularity and mixture symmetry of R are mutually independent. The axiom system used in this work entails the combination of axioms used in these two studies. There is a minor difference which is the continuity of R_0 instead of mixture continuity of R_0.

When R is a non-trivial semiorder on L, the axioms in our main result are

- R is semiorder-separable,
- R is regular,

- R is mixture-symmetric,
- R_0 is continuous, and
- R_0 satisfies midpoint indifference axiom.

We provide the following examples, where some of these examples are modified from the examples given in Dalkıran, Dokumacı, and Kara (2018), to show that these axioms are mutually independent:

Example 1. Let L be the set of lotteries on $X := \{x_1, x_2, x_3\}$ and $x, y \in L$. We define R on L as follows:

- $x\ P\ y$ if $x_1 > y_1 + 0.1$,
- $x\ I\ y$ if $|x_1 - y_1| \leq 0.1$.

Since $0.1 > 0$, it is easy to see that R is *regular*. Let D be $\mathbb{Q} \cap [0, 1]$. The set D is countably infinite and it is obviously a countable subset of L and for all $x, y \in L$ with $x\ P\ y$, there are $d_1, d_2 \in D$ such that $x\ R_0\ d_1\ P\ y$ and $x\ P\ d_2\ R_0\ y$. So, R is *semiorder-separable*. Let $x, y \in L$ and $\alpha \in (0, 1)$. Suppose $x\ I\ [\alpha x + (1\text{-}\alpha)\ y]$. This implies $|x_1 - \alpha x_1 - y_1 + \alpha y_1| \leq 0.1$. Rearranging the terms gives $|\alpha y_1 + (1 - \alpha)\ x_1 - y_1| \leq 0.1$ Hence, $y\ I\ [\alpha x + (1\text{-}\alpha)\ y]$. Thus, R is *mixture-symmetric*. For each $x,y \in L$, $x\ R_0\ y$ if and only if $x_1 \geq y_1$. Hence, R_0 is *continuous*. Let $z \in L$. Suppose for some $x,y \in L$, $x\ I_0\ y$. Because for each $x,y \in L$, $x\ I_0\ y$ if and only if $x_1 = y_1$, we have $x_1 = y_1$. Hence, $1/2x_1 + 1/2z_1 = 1/2y_1 + 1/2z_1$. Thus, $[1/2x + 1/2z]\ I_0\ [1/2y + 1/2z]$. So, it satisfies *midpoint indifference*. Therefore, *Example 1* is an example where all of our axioms hold.

Example 2. Let L be the set of lotteries on $X := \{x_1,x_2,x_3\}$ and $x,y \in L$. We define R on L as follows:

- $x\ P\ y$ if $x_1 \geq y_1 + 0.2$,
- $x\ I\ y$ if $|x_1 - y_1| < 0.2$.

Since $0.2 > 0$, it is easy to see that R is *regular*. For each $x \in L$, upper contour and lower contour sets with respect to R_0 are closed, thus R_0 is *continuous*. It is straightforward to show that for each $x, y \in L$, we have $x\ I_0\ y$ if and only if $x_1 = y_1$. Let $z \in L$ and assume that for some $p, q \in L$, we have $x\ I_0\ y$. This means $x_1 = y_1$. Thus, $1/2\ x_1 + 1/2z_1 = 1/2y_1 + 1/2z_1$, which in turn is equivalent to $[1/2\ x+1/2\ z]\ I_0\ [1/2y + 1/2z]$. So, *midpoint indifference* axiom holds. Now, the claim is this setup does not satisfy semiorder separability. To demonstrate it, suppose there exist $x, y \in L$ such that $x\ P\ y$, it means $x_1 \geq y_1 + 0.2$. And assume there is a countable subset $D \subseteq L$ with the following property: for every $x, y \in L$ such that $x\ P\ y$, there are $d_1,d_2 \in D$ such that $x\ P\ d_1\ R_0\ y$ and $x\ R_0\ d_2\ P\ y$. If $x_1 \geq y_1 + 0.2$, we will have $x\ P\ d_1\ R_0\ y$ and $x\ R_0\ d_2\ P\ y$. From former relation, $x_1 \geq d_{11} + 0.2 \geq y_1 + 0.2$ and since $x_1 = y_1 + 0.2$, we get $x_1 \geq d_{11} + 0.2 \geq x_1$ and from latter relation, $x \geq d_{21} \geq y_1 + 0.2$ and since $y_1 = x_1 - 0.2$, we get $x_1 \geq d_{21} \geq x_1$. Furthermore, $x_1 \geq d_{11} + 0.2 \geq x_1$ implies $x_1 = d_{11} + 0.2$ and $x_1 \geq d_{21} \geq x_1$ implies $x_1 = d_{21}$. These two equalities contradict with the countability of D. Therefore, R is not *semiorder separable*.

Example 3. Let L be the set of lotteries on $X := \{x_1,x_2\}$ and $x,y \in L$. We define R on L such that:

- $x\ P\ y$ if $x_1 > y_1$,
- $x\ I\ y$ if $x_1 = y_1$.

Let D be $\mathbb{Q} \cap [0, 1]$. The set D is countably infinite and it is obviously a countable subset of L and for all $x,y \in L$ with $x\ P\ y$, there are $d_1,d_2 \in D$ such that $x\ R_0\ d_1\ P\ y$ and $x\ P\ d_2\ R_0\ y$. So, R is *semiorder-separable*. It is easy to see that for each $x, y \in L$, we have $x\ R\ y$ if and only if $x\ R_0\ y$ if and only if $x_1 \geq y_1$. It implies for each $x \in L$, upper contour and lower contour sets with respect to R_0 are closed. Hence, R_0 is *continuous*. For each $x, y \in L$, we have $x\ I\ y$ if and only if $x\ I_0\ y$ if and only if $x_1 = y_1$. Let $z \in L$. Suppose for some $x, y \in L$, we have $x\ I_0\ y$. This implies $x_1 = y_1$. Hence, $1/2x_1+1/2z_1 = 1/2y_1 + 1/\ 2z_1$. Thus, $[1/2x + 1/2z]\ I_0\ [1/2y + 1/2z]$. So, *midpoint indifference* axiom holds. Since for each $x, y \in L$, $x\ P\ y$ if and only if $x_1 > y_1$, R is not *regular*.

Example 4. Let L be the set of lotteries on $X := \{x_1,x_2\}$ and $x,y \in L$. We define R on L such that:

- $x\ P\ y$ if $3x_1 > 5y_1 + 1$,
- $x\ I\ y$ if $\neg(x\ P\ y)$ and $\neg(y\ P\ x)$

$u: L \rightarrow \mathbb{R}$ as $u(x) = ln\ (x_1 + 1)$ and $k = ln\ (5/3)$ form a Scott-Suppes representation for defined R and we know that $(u, ln\ (5/3))$ is a Scott-Suppes representation of R if and only if R is *separable* and *regular*. For each $x, y \in L$, we have $x\ R\ y$ if and only if $x\ R_0\ y$ if and only if $x_1 \geq y_1$. It implies for each $x \in L$, upper contour and lower contour sets with respect to R_0 are closed. Hence, R_0 is *continuous*. For each $x, y \in L$, we have $x\ I\ y$ if and only if $x\ I_0\ y$ if and only if $x_1 = y_1$. Let $z \in L$. Suppose for some $x, y \in L$, we have $x\ I_0\ y$. This implies $x_1 = y_1$. Hence, $1/2x_1 + 1/2z_1 = 1/2y_1 + 1/2z_1$. Thus, $[1/2x_1 + 1/2z_1] = [1/2y_1 + 1/2z_1]$. So, *midpoint indifference* axiom holds. Note that for $x = (1,0)$ and $y = (0.5, 0.5)$, we have following inequalities: $3x_1 \leq 5y_1 + 1$ and $3y_1 \leq 5x_1 + 1$. Thus, $(1,0)\ I\ (0.5,0.5)$ and observe that $(0.5, 0.5) = 0.5\ (1, 0) + 0.5\ (0, 1)$ but $\neg\ ((0.5, 0.5)\ I\ (0, 1))$. Thus, R is not mixture-symmetric.

Example 5. Let L be the set of lotteries on $X := \{x_1,x_2\}$ and $x, y \in L$. We define R on L such that:

- $x\ P\ y$ if $x_1 = 1$ and $y_1 = 0$,
- $x\ I\ y$ if $x\ R\ y$ and $y\ R\ x$.

Observe that only strict preference under this setup is $(1, 0)\ P\ (0,1)$, and hence, R is trivially *separable* and *regular*. For each $x \in L$, we have $x\ I_0\ x$ and when $x_1 \in (0\ ,1)$, we have $(1, 0)\ P_0\ x\ P_0\ (0, 1)$. Accordingly, *midpoint indifference* axiom holds. To show that R_0 does not satisfy continuity consider the upper contour set of $x = (0.5, 0.5)$ with respect to R_0, i.e., $UC_0((0.5, 0.5)) = L \setminus \{(1, 0)\}$. Clearly, this set is not closed. Hence, R_0 is not *continuous*.

Example 6. Let L be the set of lotteries on $X := \{x_1,x_2\}$ and $x, y \in L$. We define R on L such that:

- $x\ P\ y$ if $x_1 > y_1 + 0.75$,
- $x\ I\ y$ if $|x_1 - y_1| \leq 0.75$.

Let D be $\mathbb{Q} \cap [0, 1]$. The set D is countably infinite and it is obviously a countable subset of L and for all $x, y \in L$ with $x P y$, there are $d_1, d_2 \in D$ such that $x R_0 d_1 P y$ and $x P d_2 R_0 y$. So, R is *semiorder-separable* and since $0.75 > 0$, it is easy to see that R is *regular*. Let $x, y \in L$ and $\alpha \in (0, 1)$. Suppose $x I (\alpha x + (1-\alpha) y)$. This implies $|x_1 - \alpha x_1 - y_1 + \alpha y_1| \leq 0.75$. Rearranging the terms gives $|\alpha y_1 + (1 - \alpha) x_1 - y_1| \leq 0.75$ Hence, $y I (\alpha x + (1 - \alpha) y)$. Thus, R is *mixture-symmetric*. For each $x, y \in L$, $x R_0 y$ if and only if $x_1 \geq y_1$. Hence, R_0 is *continuous*. Observe that $(0.75, 0.25) I_0 (0.25, 0.75)$ but $½ (0.75, 0.25) + ½ (1, 0) = (0.875, 0.125)$ and $(0.875, 0.125) P_0 (0.8, 0.2)$ where $(0.8, 0.2) = ½ (0.6, 0.3) + ½ (1, 0)$. Therefore, R_0 does not satisfy *midpoint indifference*.

Conclusion

In this paper, we focus on preferences that exhibit intransitive indifference. Many studies in the literature show that individuals either cannot recognize relatively small changes with regards to an alternative or deliberately ignore such small changes. For instance, we cannot perceive slight differences on the color scale since the eyesight of the human body has some boundaries. On the other hand, when we are about to buy something expensive like real estate, we do not attach importance to relatively small amounts in terms of prices. Such observations imply that economists should use utility representations that allow for preferences with intransitive indifference.

The main purpose of this paper is to obtain a new characterization of Expected Scott-Suppes Utility Representation (ESSUR). What makes this characterization different is that it builds upon the axioms provided by Candeal and Indurain (2010), i.e., regularity and semiorder-separability. Even though Dalkıran, Dokumacı, and Kara (2018) are the first to provide a characterization of ESSUR, their result does not use a separability axiom. Since separability axioms are essential for numerical representations of preferences on uncountable sets, it begs the answer to the question of whether a full characterization with a separability axiom is possible. By providing a new characterization of ESSUR, the main result of this study shows that the answer to this question is affirmative.

Finally, we show that the axioms we employ in our characterization are mutually independent. That is, our main result is a new full characterization of the Expected Scott-Suppes Utility Representation.

We hope that our results pave the way for future research on preferences with intransitive indifference under uncertainty.

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